IV. Conclusions

In this work, the surface integral equation has been used as a boundary condition for the finite element solution of the multiport waveguide discontinuity problem. The major advantage offered by the use of the surface integral equation approach is that it allows for placing the mesh-terminating outer boundaries of the finite element region as close to the junction discontinuity as possible, thus minimizing the size of the finite element matrix. This advantage is achieved despite the fact that the evanescent modes have significant amplitudes in the region close to the discontinuity. The accuracy of the surface integral equation formulation and its simplicity make it an efficient and versatile tool in the analysis of waveguide discontinuity problems.

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On the Use of Shanks's Transform to Accelerate the Summation of Slowly Converging Series

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***Abstract*—It is shown that the application of Shanks' transform results in accelerating the convergence of slowly converging series. The transform is applied to a periodic Green's function involving a single summation. The convergence properties of this series are reported for the "on-plane" case, in which the series converges extremely slowly. Numerical results indicate that by employing Shanks's transform the computation time can be reduced by as much as a factor of 200.**

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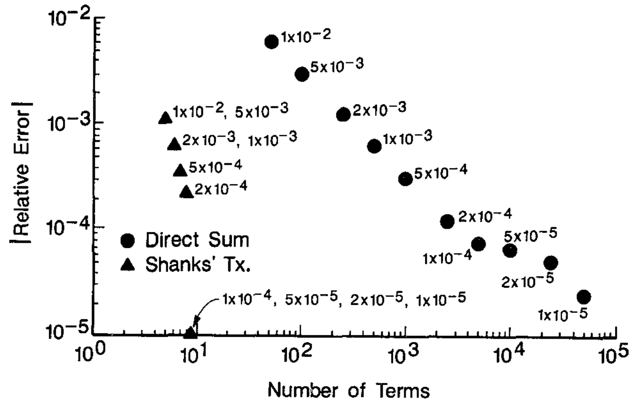

Fig. 1. Relative error magnitude versus number of terms for the series in (1) for $x = \pi/2$.

I. Introduction

In the analysis of periodic structures, one usually encounters a Green's function which converges very slowly. As repeated evaluations of the Green's function series are needed in determining the radiation or scattering from a periodic array using the method of moments with subsectionally defined basis functions, the slow convergence of the series would result in a considerable amount of computation time. In order to reduce this time, we look for ways to accelerate the convergence of the Green's function series. A method for improving the convergence of a doubly infinite periodic Green's function series has previously been suggested [1]–[3]. It has been successfully applied by a number of investigators to singly and doubly periodic Green's function series [4]–[8]. This paper reports the use of Shanks's transform in accelerating the convergence of a periodic Green's function involving a single infinite summation. Although the use of Shanks's transform in conjunction with Kummer's and Poisson's transformations has been shown in [3] to improve the convergence of a doubly periodic Green's function, it is reported here that a simple application of this transform alone to very slowly converging series enhances their convergence tremendously. Another advantage of using the transform is that no analytical work need be done to the series. This is an attractive feature, as the transform can be applied to a wide variety of series.

II. Illustrative Example of Shanks's Transform

If a sequence of partial sums of a series behaves as a "mathematical transient" as defined by Shanks in [9], then it is possible to extract the base of this "transient" by an application of Shanks's transform [9]. The transform is applied successively to the partial sums of the series until a predefined convergence criterion is satisfied. An algorithm to compute different orders of Shanks's transform is given in [10]. It is interesting to note that although the partial sums show no indication of converging to the sum of the series, the application of Shanks's transform is able to extract the sum from these partial sums. This is illustrated by taking the following series:

$$S = \sum_{n=1}^{+\infty} \frac{\sin(2n-1)x}{4\pi(2n-1)}. \tag{1}$$

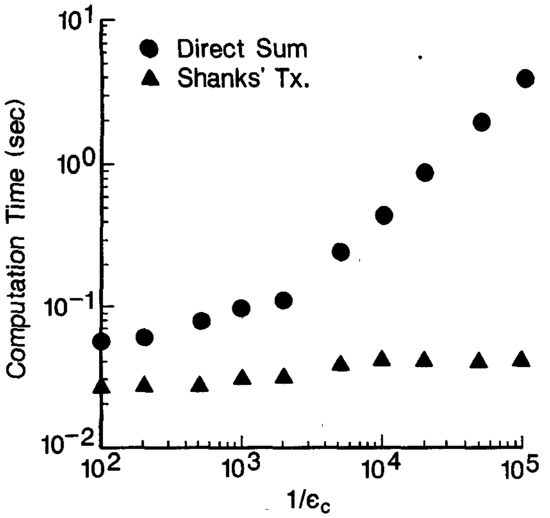

Fig. 2. Computation time versus $1/\epsilon_c$ for the series in (1) for $x = \pi/2$.

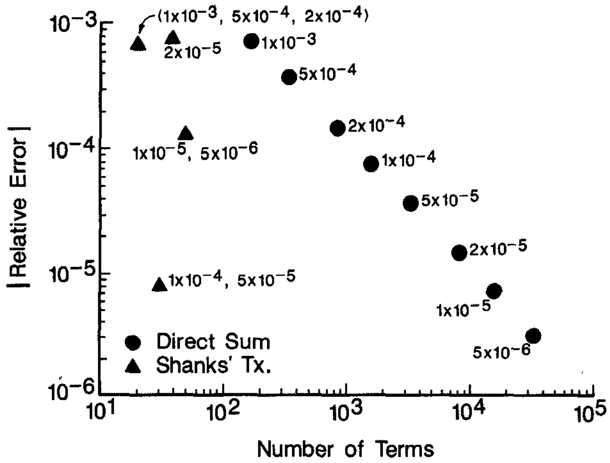

Fig. 3. Relative error magnitude versus number of terms for the periodic Green's function in (2) for $\lambda = 1$ m, $k_x = 0$, $d = 0.5\lambda$, $x' = y' = 0$, $x = 0.1\lambda$, and $y = 0$.

For $x = \pi/2$, the above series converges to 1. However, a direct summation of the series converges extremely slowly. This slow convergence is shown in Fig. 1, where the relative error is plotted as a function of the number of terms taken in the series for various values of the convergence factor, ϵ_c, given in [3]. The convergence factor is indicated alongside each point in the figure. The result obtained from a direct summation of the series is compared with that obtained from Shanks's transform. For $\epsilon_c = 10^{-4}$, Shanks's transform converges in only nine terms to an accuracy of six decimal places. On the other hand, the direct sum of the series takes about 5000 terms and converges to only five-decimal-place accuracy. Fig. 2 shows the computation time in seconds versus $1/\epsilon_c$. The computation time increases drastically as $1/\epsilon_c$ gets larger or as the convergence criterion is made more stringent. For $1/\epsilon_c = 10^5$, the computation time for the direct sum is almost a hundred times that taken by Shanks's transform.

III. Periodic Green's Function

In determining the radiation at an observation point (x, y) from a one-dimensional infinite array of line sources spaced d units apart in the x direction and located at (x', y') in each unit cell, one encounters the following Green's function [6]:

$$G = \sum_{m=-\infty}^{+\infty} \frac{1}{j2dk_{ym}} e^{-j(k_x + 2m\pi/d)(x-x')} e^{-jk_{ym}|y-y'|} \tag{2}$$

where

$$k_{ym} = \begin{cases} \sqrt{k^2 - (k_x + 2m\pi/d)^2}, & k^2 > (k_x + 2m\pi/d)^2 \\ -j\sqrt{(k_x + 2m\pi/d)^2 - k^2}, & k^2 < (k_x + 2m\pi/d)^2 \end{cases} \tag{3}$$

$$k_x = k \sin\theta \cos\phi. \tag{4}$$

Here k is the wavenumber in free space, and (θ, ϕ) are the spherical coordinate angles of an incoming or outgoing plane wave. The spectral-domain Green's function in (2) and its spatial domain counterpart, obtained by taking the inverse Fourier transform of (2), are both slowly converging. The series in (2) converges rapidly whenever $y \neq y'$. This is the so-called off-plane case, in which the exponential factor aids in the fast convergence of the series. We will look at the case in which the series converges extremely slowly. This is the on-plane case, in which $y = y'$. We set $k_x = 0$, $d = 0.5\lambda$, $x' = y' = 0$, $y = 0$, and $\lambda = 1$ m, and we will consider values of x close to x', specifically $x = 0.1\lambda$ and $x = 0.01\lambda$. In order to compute the relative error, the series in (2) is first summed to machine precision. This sum is then used in calculating the relative error.

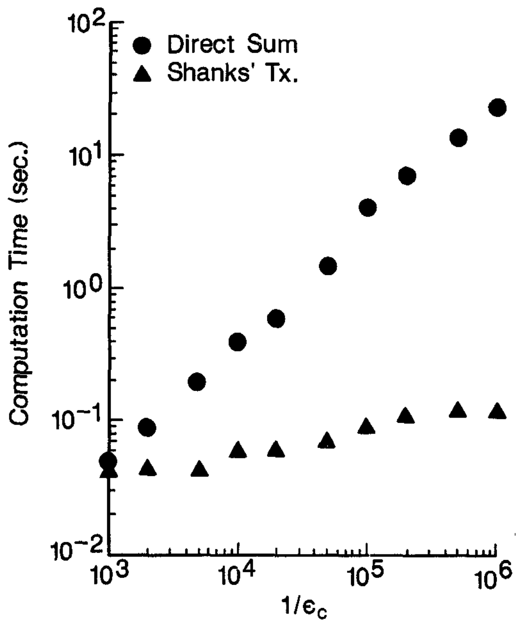

Fig. 4. Computation time versus $1/\epsilon_c$ for the periodic Green's function in (2) for $\lambda = 1$ m, $k_x = 0$, $d = 0.5\lambda$, $x' = y' = 0$, $x = 0.1\lambda$, and $y = 0$.

Fig. 3 shows the relative error magnitude versus the number of terms for various values of ϵ_c for $x = 0.1\lambda$. The dramatic improvement in the rate of convergence of the series due to Shanks's transform is illustrated in this figure. The transform converges in fewer than 50 terms for the range of ϵ_c. However, the direct sum converges slowly, taking as many as 33 000 terms for $\epsilon_c = 5 \times 10^{-6}$. The computation time in seconds versus $1/\epsilon_c$ for this case is shown in Fig. 4. The time taken by Shanks's transform varies from 0.04 to 0.12 s, while that for the direct

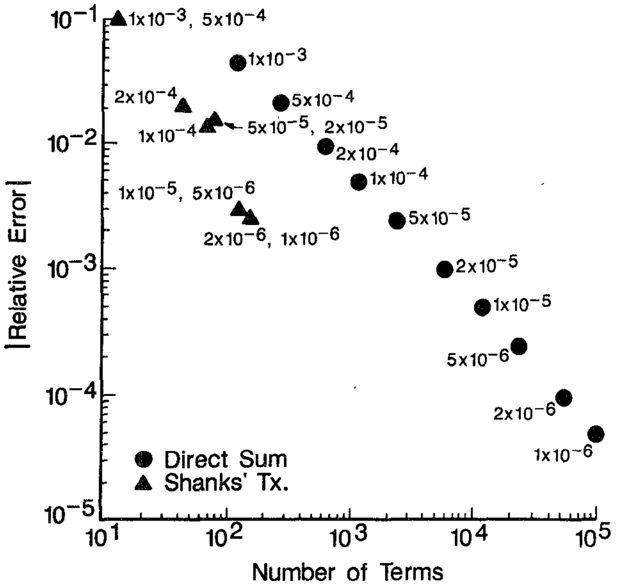

Fig. 5. Relative error magnitude versus number of terms for the periodic Green's function in (2) for $\lambda = 1$ m, $k_x = 0$, $d = 0.5\lambda$, $x' = y' = 0$, $x = 0.01\lambda$, and $y = 0$.

sum varies from 0.05 to 24 s. This translates into a saving factor ranging from 1.25 to 200 in using the transform.

Next, the observation point is brought closer to the source point; that is, we take $x = 0.01\lambda$. For this case the series converges much slower than in the $x = 0.1\lambda$ case. Fig. 5 shows the relative error magnitude versus the number of terms. The result obtained from Shanks's transform converges within 160 terms; meanwhile the direct sum requires as many as 100000 terms for $\epsilon_c = 1 \times 10^{-6}$. The computation time for this case (although not shown graphically) varies from 0.11 to 17.38 s for the direct sum and 0.04 to 0.61 s for the transform. In this case a saving factor of 2.75 to 30 is obtained by employing the transform. It should be pointed out that for series that converge at a very slow rate, the convergence factor should be of the order of 10^{-5} or lower. This is to ensure that the summing process does not stop prematurely due to "local" convergence.

IV. Conclusions

It is shown here, with an example involving two very slowly converging series, that the application of Shanks's transform accelerates dramatically the convergence of such series. This is indicated by the computation time taken by Shanks's transform, which in some cases is reduced by a factor of 200 compared with that taken by the direct summation of the series. The transform seems to work particularly well for very slowly converging series where the partial sums of the series exhibit an oscillatory behavior.

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